The second term is given by (12.28). The first term equals

$$-\left[X^{J\prime}(\tau,\sigma'),\,\dot{X}^I(\tau,\sigma)\right] = -(2\pi\alpha')i\eta^{JI}\frac{d}{d\sigma'}\delta(\sigma'-\sigma) = 2\pi\alpha'\,i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma'). \tag{12.32}$$

To obtain this result we noted that a σ' derivative can be traded for minus a σ derivative when it acts on a function of $(\sigma-\sigma')$. Moreover, we used $\delta(x)=\delta(-x)$. We now see that both terms in (12.31) are equal, so

$$\left[(\dot{X}^I+X^{I\prime})(\tau,\sigma),\,(\dot{X}^J+X^{J\prime})(\tau,\sigma')\right] = 4\pi\alpha'\,i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma'). \tag{12.33}$$

In fact, more generally, we have found that

$$\boxed{\left[(\dot{X}^I\pm X^{I\prime})(\tau,\sigma),\,(\dot{X}^J\pm X^{J\prime})(\tau,\sigma')\right] = \pm 4\pi\alpha'\,i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma'),} \tag{12.34}$$

since only the cross terms contribute. Finally,

$$\left[(\dot{X}^I\pm X^{I\prime})(\tau,\sigma),\,(\dot{X}^J\mp X^{J\prime})(\tau,\sigma')\right] = 0. \tag{12.35}$$

Equations (12.34) and (12.35) hold for $\sigma,\sigma'\in[0,\pi]$.

We consider now the commutator of the function defined in (12.26) at σ, with the same function at σ'. If $\sigma,\sigma'\in[0,\pi]$ we can use the top right-hand side of (12.26) together with (12.33) to find

$$\begin{aligned}2\alpha'\sum_{m',n'\in\mathbb{Z}} e^{-im'(\tau+\sigma)}e^{-in'(\tau+\sigma')}[\alpha^I_{m'},\alpha^J_{n'}] &= \left[(\dot{X}^I+X^{I\prime})(\tau,\sigma),\,(\dot{X}^J+X^{J\prime})(\tau,\sigma')\right]\\ &= 4\pi\alpha'\,i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma').\end{aligned}$$

Cancelling the common factor of $2\alpha'$,

$$\sum_{m',n'\in\mathbb{Z}} e^{-im'(\tau+\sigma)}e^{-in'(\tau+\sigma')}[\alpha^I_{m'},\alpha^J_{n'}] = 2\pi\,i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma'). \tag{12.36}$$

We now claim that this equation actually holds for $\sigma,\sigma'\in[-\pi,\pi]$. If $\sigma\in[-\pi,0]$ and $\sigma'\in[0,\pi]$, or vice versa, the commutator we get is that in (12.35), which vanishes. Accordingly, the right-hand side of (12.36) also vanishes since in this case σ and σ' cannot be equal. When both $\sigma,\sigma'\in[-\pi,0]$, the commutator is

$$\left[(\dot{X}^I-X^{I\prime})(\tau,-\sigma),\,(\dot{X}^J-X^{J\prime})(\tau,-\sigma')\right], \tag{12.37}$$

and making use of (12.34), this equals

$$= -4\pi\alpha'\,i\eta^{IJ}\frac{d}{d(-\sigma)}\delta(-\sigma+\sigma') = 4\pi\alpha'\,i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma'), \tag{12.38}$$

which coincides with the result that led to (12.36). We have therefore shown that (12.36) holds for $\sigma,\sigma'\in[-\pi,\pi]$.