Problem 1

Question 1

$[x^+, p^-]$

* In l.c. gauge $\quad x^+ = \frac{p^+}{m^2}\tau \qquad p^- = \frac{1}{2p^+}(p^I p^I + m^2)$

$$[x^+, p^-] = 0$$

* In covariant quantization $\quad [x^+, p^-] = i\eta^{+-} = -i$

$\boxed{\text{correct answer is (d)}}$

Question 2

(a) X^- satisfies NN b.c

(b) The periodicity of X^- implies $L_0^\perp = \bar{L}_0^\perp$

(c) $\quad p^\mu = (M, \vec{0}) \qquad p^0 = M$

$p^1 = 0$

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^1) = \frac{M}{\sqrt{2}}$$

Question 3

(a) True (P generates $\sigma \to \sigma +$ constant)

(b) True $\quad P^+ = P$ since $(L_0^\perp)^+ = L_0^\perp$

$(\bar{L}_0^\perp)^+ = \bar{L}_0^\perp$

(c) False, only $P=0$ states are allowed

(d) True $\quad [P, L_0^\perp + \bar{L}_0^\perp] = 0$

Question 4

(a) $|R_a\rangle_L \otimes |R_b\rangle_R \otimes |p^+, \vec{p}_T\rangle$

RR sector
64 bosons
$M^2 = 0$

(b) $b^I_{-1/2}\, \bar{b}^J_{-1/2}\; |NS\rangle_L \otimes |NS\rangle_R \otimes |p^+, \vec{p}_T\rangle$

64 bosons
$M^2 = 0$

NS-NS sector (graviton, KR + dilaton)

(c) $\bar{b}^I_{-1/2}\, |NS\rangle_L \otimes |R_a\rangle_R \otimes |p^+, p_T\rangle$

64 fermions
$M^2 = 0$

NS-R sector

Question 5

	$\alpha' M_L^2$		$\alpha' M_R^2$
$\lvert NS\rangle_L$	$-\frac{1}{2}$	$\lvert NS\rangle_R$	$-\frac{1}{2}$
$\underbrace{\bar{\alpha}^I_{-1}\lvert NS\rangle_L \;(8), \quad \bar{b}^I_{-1/2}\bar{b}^J_{-1/2}\lvert NS\rangle_L \;(\frac{8\times 7}{2} = 28)}_{36\text{ states}}$	$+\frac{1}{2}$	$\underbrace{\alpha^K_{-1}\lvert NS\rangle_R, \; b^P_{-\frac{1}{2}} b^Q_{-\frac{1}{2}}\lvert NS\rangle_R}_{36\text{ states}}$	$+\frac{1}{2}$

Closed string:

$|NS\rangle_L \otimes |NS\rangle_R$ $\qquad \boxed{\alpha' M_{cl}^2 = -2}$

$\bar{\alpha}^I_{-1}\alpha^J_{-1}$ $\quad 8\times 8$

$\bar{\alpha}^I_{-1} b^J_{-\frac{1}{2}} b^K_{-\frac{1}{2}}$ $\quad$ on $|NS\rangle_L \otimes |NS\rangle_R$ $\quad 8\times 28$

$\bar{b}^I_{-1/2}\bar{b}^J_{-1/2}\alpha^K_{-1}$ $\quad 28\times 8$

$\bar{b}^I_{-\frac{1}{2}}\bar{b}^J_{-\frac{1}{2}}\, b^P_{-\frac{1}{2}} b^Q_{-\frac{1}{2}}$ $\quad 28\times 28$

36×36 states
$= 1296$ states

$\boxed{\alpha' M_{cl}^2 = +2}$

Problem 2

Equation of motion

$$\partial^\rho \left(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \right) = 0$$

in momentum space

$$p^\rho \left(p_\mu B_{\nu\rho} + p_\nu B_{\rho\mu} + p_\rho B_{\mu\nu} \right) = 0$$

$$\boxed{p^2 B_{\mu\nu} = - p_\mu p^\rho B_{\nu\rho} - p_\nu p^\rho B_{\rho\mu}} \quad \text{(I)}$$

Gauge transformations: $\partial_\mu \Rightarrow i p_\mu$

$$\delta B_{\mu\nu}(p) = i p_\mu \epsilon_\nu - i p_\nu \epsilon_\mu \quad \text{--- (2)}$$

Write (I) as:

$$p^2 B_{\mu\nu}(p) = i p_\mu (i p^\rho B_{\nu\rho}) - i p_\nu (-i p^\rho B_{\rho\mu})$$

$$= i p_\mu (i p^\rho B_{\nu\rho}) - i p_\nu (i p^\rho B_{\mu\rho})$$

Therefore

$$B_{\mu\nu}(p) = i p_\mu \left(\frac{i p^\rho B_{\nu\rho}}{p^2} \right) - i p_\nu \left(\frac{i p^\rho B_{\mu\rho}}{p^2} \right)$$

$$= i p_\mu \chi_\nu - i p_\nu \chi_\mu \,, \quad \text{with } \chi_\mu = \frac{i p^\rho B_{\mu\rho}}{p^2}$$

Thus $B_{\mu\nu}(p)$ is pure gauge when $p^2 \neq 0$

Problem 3

(a)
$$L^{\perp}_{-4} = \frac{1}{2}\left(\alpha^I_{-2}\alpha^I_{-2} + \alpha^I_{-3}\alpha^I_{-1} + \alpha^I_{-4}\alpha^I_{0} + \alpha^I_{-5}\alpha^I_{1} + \cdots\right.$$
$$\left. + \alpha^I_{-1}\alpha^I_{-3} + \alpha^I_{0}\alpha^I_{-4} + \alpha^I_{1}\alpha^I_{-5} + \cdots\cdots\right)$$

$$L^{\perp}_{-4} = \frac{1}{2}\alpha^I_{-2}\alpha^I_{-2} + \alpha^I_{-3}\alpha^I_{-1} + \cdots \text{ terms that kill } |0\rangle$$

$$\boxed{L^{\perp}_{-4}|0\rangle = \left(\frac{1}{2}\alpha^I_{-2}\alpha^I_{-2} + \alpha^I_{-3}\alpha^I_{-1}\right)|0\rangle}$$

(b)
$$L^{\perp}_{+4} = \frac{1}{2}\alpha^I_{2}\alpha^I_{2} + \alpha^I_{3}\alpha^I_{1} + \underbrace{\alpha^I_{4}\alpha^I_{0} + \alpha^I_{-1}\alpha^I_{5} + \cdots}_{\text{all these kill } L^{\perp}_{-4}|0\rangle}$$

$$L^{\perp}_{+4}L^{\perp}_{-4}|0\rangle = \left(\frac{1}{2}\alpha^J_{2}\alpha^J_{2} + \alpha^J_{3}\alpha^J_{1}\right)\left(\frac{1}{2}\alpha^I_{-2}\alpha^I_{-2} + \alpha^I_{-1}\alpha^I_{-3}\right)|0\rangle$$
$$= \frac{1}{4}\alpha^J_{2}\underbrace{\alpha^J_{2}\alpha^I_{-2}\alpha^I_{-2}}|0\rangle + \alpha^J_{3}\alpha^J_{1}\alpha^I_{-1}\alpha^I_{-3}|0\rangle$$
$$= \frac{1}{4}\alpha^J_{2}\, 2\cdot 2\,\alpha^I_{-2}\delta^{IJ}|0\rangle + \alpha^J_{3}\alpha^J_{-3}|0\rangle$$
$$= \alpha^J_{2}\alpha^J_{-2}|0\rangle + 3(D-2)|0\rangle$$

$$\boxed{L^{\perp}_{+4}L^{\perp}_{-4}|0\rangle = 5(D-2)|0\rangle}$$

From Virasoro
$$[L^{\perp}_{4}, L^{\perp}_{-4}] = 8L^{\perp}_{0} + \frac{1}{12}4(16-1)(D-2)$$
$$= 8L^{\perp}_{0} + \underline{5(D-2)}$$

fine since $L^{\perp}_{0}|0\rangle = 0$!

(a) $[L_n^\perp, N^\perp]$

Recall $L_0^\perp = \alpha p^I p^I + N^\perp$

$$[L_n^\perp, N^\perp] = \underbrace{[L_n^\perp, -\alpha p^I p^I]}_{0} + [L_n^\perp, L_0^\perp]$$

$$\boxed{[L_n^\perp, N^\perp] = n L_n^\perp}$$

(c) Let $|\chi\rangle$ be annihilated by $L_1^\perp$, and $L_2^\perp$

$$L_1^\perp |\chi\rangle = 0$$

$$L_2^\perp |\chi\rangle = 0$$

Note that then it follows that

$$\underbrace{[L_2^\perp, L_1^\perp]}_{\text{using Virasoro als}} |\chi\rangle = 0 \quad \left(\begin{array}{l} \text{indeed:} \\ L_2^\perp L_1^\perp |\chi\rangle - L_1^\perp L_2^\perp |\chi\rangle \\ = L_2^\perp(0) - L_1^\perp(0) = 0 \end{array} \right)$$

$$L_3^\perp |\chi\rangle = 0$$

Clearly, by induction, if $L_n^\perp |\chi\rangle = 0$ for $n = 1, \ldots N$ $(N > 1)$
then

$$L_{N+1}^\perp |\chi\rangle = \frac{1}{(N-1)} [L_N^\perp, L_1^\perp] |\chi\rangle = 0$$

Thus, $|\chi\rangle$ is annihilated by all positively moded Virasoro operators.

Problem 4 Designing on R sector

$X^i \qquad i = 1 \ldots k \qquad \alpha^i_n$

$\lambda^A \qquad A = 1, \ldots, 2l \qquad \lambda^A_n$

(a) Naive mass formula:

$$\alpha' M^2 = \frac{1}{2}\sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n\neq 0} n\lambda^A_{-n}\lambda^A_n$$

Then:

$$\alpha' M^2 = \sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_{+n} + n\lambda^A_{-n}\lambda^A_n\right) + k\left(-\frac{1}{24}\right) + 2l\left(\frac{1}{24}\right)$$

using $a_B = -\frac{1}{24} \quad a_R = +\frac{1}{24}$

So we get

$$\boxed{\alpha' M^2 = \sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + n\lambda^A_{-n}\lambda^A_n\right) + \frac{1}{24}(2l-k)}$$

(b) $2l$, λ^A_0 - zero modes $\rightarrow$ l creation

$\rightarrow \boxed{2^l \text{ ground states}}$

(c)

$$\boxed{f_R(x) = 2^l\, x^{\frac{1}{24}(2l-k)} \prod_{n=1}^{\infty} \frac{(1+x^n)^{2l}}{(1-x^n)^k}}$$

Problem 5

(a) $\#\text{sectors} = 2^{16} = 65{,}536$

$$U_a X^b(\tau,\sigma) U_a^{-1} = \begin{cases} X^b(\tau,\sigma) & \text{if } b \neq a \\ -X^b(\tau,\sigma) & \text{if } b = a \end{cases}$$

$$\left(\text{or} \quad (-1)^{\delta_{ab}} X^b(\tau,\sigma)\right)$$

$$U_a X^i(\tau,\sigma) U_a^{-1} = X^i(\tau,\sigma)$$

(b) Naive

$$\alpha' M_L^2 = \frac{1}{2}\sum_{n\neq 0} \bar\alpha^i_{-n}\bar\alpha^i_n + \frac{1}{2}\sum_{r\in \mathbb{Z}+\frac{1}{2}} \bar\alpha^a_{-r}\bar\alpha^a_r$$

The antiperiodic ordering constant is $+\frac{1}{48}$. Indeed

$$\frac{1}{2}\left(\bar\alpha^a_{\frac{1}{2}}\bar\alpha^a_{-\frac{1}{2}} + \bar\alpha^a_{\frac{3}{2}}\bar\alpha^a_{-\frac{3}{2}} + \cdots\cdots\right)$$

$$= \text{well ordered} + \frac{1}{2}\left(\frac{1}{2}+\frac{3}{2}+\cdots\right)(16)$$

$$= \frac{1}{4}(1+3+5+7+\ldots)16 = \frac{1}{4}\frac{1}{12}\cdot 16 = \frac{1}{48}\cdot 16$$

So

$$\alpha' M_L^2 = \sum_{n=1}^{\infty}\bar\alpha^i_{-n}\bar\alpha^i_n + \sum_{r=\frac{1}{2},\frac{3}{2},} \bar\alpha^a_{-r}\bar\alpha^a_r \underbrace{-\frac{8}{24}+\frac{16}{48}}_{\text{cancel!}}$$

$$\boxed{\alpha' M_L^2 = \sum_{n=1}^{\infty}\bar\alpha^i_{-n}\bar\alpha^i_n + \sum_{r=\frac{1}{2},\frac{3}{2},} \bar\alpha^a_{-r}\bar\alpha^a_r}$$

(c)

$$\boxed{f_L(x) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^8} \frac{1}{(1-x^{n-\frac{1}{2}})^{16}}}$$

d) There are no zero modes in the $X^a(\tau,\sigma)$ in the twisted sector

ground states $|p^+, p^1, p^2, \dots p^9\rangle \equiv |p^+, p^i\rangle$

e)

Consider the left sector oscillators on

	$\alpha' M_L^2$	# states
1	0	1
$\bar\alpha^a_{-\frac{1}{2}}$	$\frac{1}{2}$	16
$\bar\alpha^a_{-\frac{1}{2}} \bar\alpha^b_{-\frac{1}{2}}$, $\bar\alpha^i_{-1}$	1	$8 + \frac{16 \cdot 17}{2} = 8 + 136 = 144$

U_a invariant

$\bar\alpha^a_{-\frac{1}{2}} \bar\alpha^b_{-\frac{1}{2}} \rightarrow \underbrace{\bar\alpha^a_{-\frac{1}{2}} \bar\alpha^a_{-\frac{1}{2}}}$ + $\bar\alpha^a_{-\frac{1}{2}} \bar\alpha^b_{-\frac{1}{2}}$ $a \neq b$

16 of these, U invariant (a not summed)

$\frac{16 \times 15}{2} = 120$ of these

So actually at the third level the 144 states split into 24 that are fully U-invariant, 120 that are not

$$\frac{1}{2} \alpha' M_{cl} = \alpha' M_L^2 + \alpha' M_R^2$$

To produce closed string states we tensor these against analogous right movers

	$\alpha' M_{cl}^2$	# states
$\lvert p^+, p^i\rangle$	0	1
$\bar\alpha^a_{-\frac12}\,\alpha^a_{-\frac12}\lvert p^+, p^i\rangle$	2	16

(note that $\bar\alpha^a_{-\frac12}\,\alpha^b_{-\frac12}\lvert p^+, p^i\rangle$ is not U invariant for $a \neq b$)

$\bar\alpha^i_{-1}\,\alpha^j_{-1}\lvert p^+, p^i\rangle$	4	8×8
$\bar\alpha^i_{-1}\,\alpha^a_{-\frac12}\alpha^a_{-\frac12}\lvert p^+, p^i\rangle$		8×16
$\bar\alpha^a_{-\frac12}\bar\alpha^a_{-\frac12}\,\alpha^i_{-1}\lvert p^+ p^i\rangle$		16×8
$\bar\alpha^a_{-\frac12}\bar\alpha^a_{-\frac12}\,\alpha^b_{-\frac12}\alpha^b_{-\frac12}\lvert p^+, p^i\rangle$		16×16
$\bar\alpha^a_{-\frac12}\bar\alpha^b_{-\frac12}\,\alpha^a_{-\frac12}\alpha^b_{-\frac12}\lvert p^+, p^i$		120

$(24)^2 = 576$

total 696

(e) $\lvert p^+, p^i\rangle$ get wavefunctions $\psi(\tau, p^+, p^i)$

or in coordinate space $\psi(x^+, x^-, x^i)$ $\quad i = 2, \dots 9$

$\psi(X^\mu)$ $\quad \mu = 0, \dots 9$ $\quad$ <u>10 dim fields</u>

live at the orbifold plane $X^a = 0$ $\quad a = 1 \dots 16$