8.251 Test 2

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Only personal 2-page notes allowed.
Formula sheet on the last page.
Test duration: 1.5 hours.

Problem 1. 5 short questions (20 points; 4 points each)

Question 1. Consider the commutator $[x^+, p^-]$ in the quantization of the point particle. Which of the following statements is true?

(a) It vanishes both in light-cone gauge quantization and in covariant quantization.

(b) It is non-zero both in light-cone gauge quantization and in covariant quantization.

(c) It is non-zero in light-cone gauge quantization but vanishes in covariant quantization.

(d) It vanishes in light-cone gauge quantization but is it non-zero in covariant quantization.

Question 2. Strings in the light-cone gauge.

(a) What boundary conditions does the open string coordinate X^- satisfy? Circle the correct answer among:

NN ND DN DD

(b) What condition does the closed string periodicity $X^-(\tau, \sigma) = X^-(\tau, \sigma + 2\pi)$ imply on the closed string Virasoro operators?

(c) A string of mass M has zero spatial momentum. What is the corresponding value of p^+?

Question 3. Consider the following statements concerning the closed string operator $P = L_0^\perp - \bar{L}_0^\perp$. Write a T to the left of the true statements and an F to the left of the false statements.

(a) P generates spatial translations on the world-sheet.

(b) P is hermitian.

(c) Any eigenstate of P is an allowed closed string state.

(d) P commutes with the closed string Hamiltonian.

Question 4. Consider the following states of type II superstrings. Indicate the sector that they belong to, how many states there are, their masses, and if they are (spacetime) bosons or fermions.

(a) $|\mathrm{R}_a\rangle_L \otimes |\mathrm{R}_b\rangle_R \otimes |p^+, \vec{p}_T\rangle$.

(b) $b^I_{-1/2}\, \bar{b}^J_{-1/2} |\mathrm{NS}\rangle_L \otimes |\mathrm{NS}\rangle_R \otimes |p^+, \vec{p}_T\rangle$.

(c) $\bar{b}^I_{-1/2}\, |\mathrm{NS}\rangle_L \otimes |\mathrm{R}_a\rangle_R \otimes |p^+, \vec{p}_T\rangle$.

Question 5. Construct the first two mass levels (ground and first excited) of the closed string sector (NS−, NS−). List the states, give the number of them, and their masses.

Problem 2. (10 points) Trivial Kalb-Ramond fields.

For a Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$ the equation of motion is

$$\partial^\rho H_{\mu\nu\rho} = 0\,, \quad \text{with} \quad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\,,$$

and the gauge transformations are

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu\,.$$

Examine the above in momentum space and show that field configurations $B_{\mu\nu}(p)$ that satisfy the equation of motion are pure gauge for $p^2 \neq 0$.

Problem 3. (20 points) Questions on Virasoro operators.

(a) Express the open string state

$$L_{-4}^{\perp}|0\rangle$$

in terms of normal-ordered oscillators acting on the zero-momentum ground state $|0\rangle$.

(b) Calculate explicitly the action of $L_4^{\perp}$ on the state found in (a), in other words, simplify the open string state $L_4^{\perp} L_{-4}^{\perp}|0\rangle$. Confirm your answer using the Virasoro algebra.

(c) Prove that any state $|\chi\rangle$ annihilated by $L_1^{\perp}$ and $L_2^{\perp}$ is also annihilated by all the positively moded Virasoro operators.

(d) Calculate the commutator $[L_n^{\perp}, N^{\perp}]$.

Problem 4 (20 points) Designing a Ramond sector.

The Ramond sector of an open string theory has k transverse bosonic coordinates X^i $(i = 1, \ldots, k)$ with associated oscillators α_n^i of familiar commutation relations and 2ℓ fermionic fields λ^A $(A = 1, \ldots, 2\ell)$. These periodic fermions, as befits the Ramond sector, and have integrally moded oscillators λ_n^A with anticommutation relations $\{\lambda_m^A, \lambda_n^B\} = \delta_{m+n,0}\delta^{AB}$.

(a) Write a precise mass formula in terms of well-ordered products of oscillators and the appropriate ordering constant.

(b) What is the vacuum degeneracy ?

(c) Write the function $f_R(x)$ for which the coefficient of x^k is the number of states with $\alpha' M^2 = k$.

Problem 5 (30 points) Counting states of closed strings on the $(\mathbb{R}/\mathbb{Z}_2)^{16}$ orbifold.

Consider the closed bosonic string in which 16 spatial dimensions x^a, with $a = 10, \ldots, 25$ are independently orbifolded to form half-lines $x^a \sim -x^a$. The corresponding string coordinates are called $X^a(\tau, \sigma)$ and the rest are X^+, X^- and X^i with i taking the eight values $i = 2, \ldots, 9$.

Recall that each time that we produce a half-line, we get two sectors: untwisted with integrally moded operators and twisted with half-integrally moded oscillators.

(a) What is the total number of sectors that the string theory on $(\mathbb{R}/\mathbb{Z}_2)^{16}$ has ? The theory has 16 operators U_a that implement the orbifold symmetry. Complete the equations

$$U_a X^b(\tau, \sigma)\, U_a^{-1} = \ldots \ ,$$

$$U_a X^i(\tau, \sigma)\, U_a^{-1} = \ldots \ ,$$

Consider the sector of the closed string theory that is twisted in all 16 coordinates. The *naive* mass formula for the left-moving part reads

$$\alpha' M_L^2 = \frac{1}{2} \sum_{n \neq 0} \bar{\alpha}^i_{-n} \bar{\alpha}^i_n + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \bar{\alpha}^a_{-r} \bar{\alpha}^a_r \, .$$

A completely similar formula with unbarred oscillators exists for the right-moving part.

(b) Determine the ordering constant needed and then write a precise formula for $\alpha' M_L^2$.

(c) Write a generating function $f_L(x) = \sum_k a(k)\, x^k$ such that $a(k)$ is the number of states in the left-moving sector with $\alpha' M_L^2 = k$.

(d) What are the momentum labels of the closed string ground states?

(e) Construct explicitly the closed string states for the first three mass levels (ground states plus two more). Indicate the value of $\alpha' M^2_{\text{closed}}$ and count the number of states at each level. Recall that all closed string states must be invariant under every U_a operator.

(e) There are fields associated with the states above. What coordinates do they depend on?

Possibly Useful Formulas

Light-Cone Coordinates: $x^{\pm} = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right)$.

Relativistic Point Particle in Light-Cone Coordinates: $x^+ = \frac{p^+}{m^2}\,\tau$, $\quad p^- = \frac{1}{2p^+}\left(p^I p^I + m^2\right)$.

Slope Parameter: $\alpha' = \frac{1}{2\pi T_0}$.

Light-Cone Gauge:

$$X^+ = \beta\alpha' p^+ \tau, \quad \text{where} \quad \beta = \begin{cases} 2 & \text{for open strings} \\ 1 & \text{for closed strings} \end{cases},$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \; , \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu\prime} \; ,$$

$$(\dot{X} \pm X')^2 = 0 \quad \Longrightarrow \quad \dot{X}^- \pm X^{-\prime} = \frac{1}{2\beta\alpha' p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2 \; ,$$

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0 \; .$$

Open String Expansion: $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-in\tau}\,\cos n\sigma$.

Closed String Expansion: $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^\mu\, e^{in\sigma} + \bar{\alpha}_n^\mu\, e^{-in\sigma}\right)$.

Momentum: $\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu$ (open strings) , $\qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu$ (closed strings).

Commutators, Creation and Annihilation Operators:

$$\left[\alpha_n^I\, ,\, \alpha_m^J\right] = n\delta_{m+n,0}\,\delta^{IJ} \; , \qquad \left[x_0^I\, ,\, \alpha_0^J\right] = \sqrt{2\alpha'}\, i\,\delta^{IJ} \; , \qquad \left[x_0^I\, ,\, \alpha_n^J\right] = 0 \text{ if } n \neq 0 \; ,$$

$$\text{for } n \geq 1: \quad \alpha_n^I = \sqrt{n}\, a_n^I \; , \qquad \alpha_{-n}^I = \sqrt{n}\, a_n^{I\dagger} \; , \qquad \left[a_m^I\, ,\, a_n^{J\dagger}\right] = \delta^{IJ}\,\delta_{mn} \; .$$

Virasoro Algebra (Open Strings):

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\,p^+}L_n^\perp \; , \qquad \text{where} \quad L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I \; , \quad \left[L_m^\perp\, ,\, \alpha_n^J\right] = -n\alpha_{m+n}^J \; ,$$

$$\left[L_m^\perp\, ,\, L_n^\perp\right] = (m-n)L_{m+n}^\perp + \frac{1}{12}m(m^2-1)(D-2)\delta_{m+n,0} \; , \quad \left[L_m^\perp\, ,\, x_0^I\right] = -i\sqrt{2\alpha'}\,\alpha_m^I \; ,$$

$$L_0^\perp = \alpha'\,p^I\,p^I + N^\perp \; , \qquad N^\perp = \sum_{p\neq 0}\alpha_{-p}^I\,\alpha_p^I = \sum_{n=1}^{\infty} n\, a_n^{I\dagger}\, a_n^I \; ,$$

$$M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I = \frac{1}{\alpha'}\left(N^\perp - 1\right) \; .$$

Virasoro Algebra (Closed Strings):

$$\alpha_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\,L_n^\perp \; , \quad \bar{\alpha}_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\,\bar{L}_n^\perp \; , \quad L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I \; , \quad \bar{L}_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\bar{\alpha}_{n-p}^I\,\bar{\alpha}_p^I \; ,$$

$$L_0^\perp = \frac{\alpha'}{4}\,p^I\,p^I + N^\perp \; , \quad N^\perp = \sum_{p\neq 0}\alpha_{-p}^I\,\alpha_p^I = \sum_{n=1}^{\infty} n\, a_n^{I\dagger}\, a_n^I \; ,$$

$$\bar{L}_0^\perp = \frac{\alpha'}{4}\,p^I\,p^I + \bar{N}^\perp \; , \quad \bar{N}^\perp = \sum_{p\neq 0}\bar{\alpha}_{-p}^I\,\bar{\alpha}_p^I = \sum_{n=1}^{\infty} n\, \bar{a}_n^{I\dagger}\, \bar{a}_n^I \; ,$$

$$M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I = \frac{2}{\alpha'}\left(N^\perp + \bar{N}^\perp - 2\right) \; , \quad \bar{N}^\perp = N^\perp \; .$$

NS-sector:

Ground state: $(-1)^F = -1$: $|\text{NS}\rangle$.

Oscillators: α_n^I, b_r^I, with $\{b_r^I, b_s^J\} = \delta_{r+s,0}\delta^{IJ}$.

Normal ordering constant for NS fermion: $a_{\text{NS}} = -\frac{1}{48}$.

Mass-squared: $\alpha' M^2 = -\frac{1}{2} + N^\perp$.

$$
\begin{aligned}
\alpha' M^2 = -\tfrac{1}{2}\,,\ N^\perp = 0 &: |\text{NS}\rangle\,,\\
\alpha' M^2 = \ 0\,,\ N^\perp = \tfrac{1}{2} &: b^I_{-1/2}|\text{NS}\rangle\,,\\
\alpha' M^2 = \ \tfrac{1}{2}\,,\ N^\perp = 1 &: \big\{\alpha^I_{-1}\,,\ b^I_{-1/2}b^J_{-1/2}\big\}|\text{NS}\rangle\,,\\
\alpha' M^2 = 1\,,\ N^\perp = \tfrac{3}{2} &: \big\{\alpha^I_{-1}b^J_{-1/2}\,,\ b^I_{-3/2}\,,\ b^I_{-1/2}b^J_{-1/2}b^K_{-1/2}\big\}|\text{NS}\rangle\,.
\end{aligned}
$$

NS+ sector: $(-1)^F = +1$. Integer $\alpha' M^2$.

R-sector:

8 zero modes d_0^I: $\to 4\,$creation $+\,4$ annihilation. 4 creation $\to 2^4 = 16$ ground states.

Ground states: $(-1)^F = -1$: $|R_a\rangle$, $a = 1, \dots 8$, and $(-1)^F = +1$: $|R_{\bar a}\rangle$, $\bar a = \bar 1, \dots \bar 8$.

Oscillators: α_n^I, d_n^I, with $\{d_m^I, d_n^J\} = \delta_{m+n,0}\delta^{IJ}$.

Normal ordering constant for Ramond fermion: $a_{\text{R}} = +\frac{1}{24}$.

Mass-squared: $\alpha' M^2 = N^\perp$.

$$
\begin{array}{lr||l}
\alpha' M^2 = 0: & |R_a\rangle & |R_{\bar a}\rangle\\
\alpha' M^2 = 1: & \alpha^I_{-1}|R_a\rangle\,,\ d^I_{-1}|R_{\bar a}\rangle & \alpha^I_{-1}|R_{\bar a}\rangle\,,\ d^I_{-1}|R_a\rangle,\\
\alpha' M^2 = 2: & \{\alpha^I_{-2}\,,\ \alpha^I_{-1}\alpha^J_{-1}\,, d^I_{-1}d^J_{-1}\}|R_a\rangle & \{\alpha^I_{-2}\,,\ \alpha^I_{-1}\alpha^J_{-1}\,, d^I_{-1}d^J_{-1}\}|R_{\bar a}\rangle\\
 & \{\alpha^I_{-1}d^J_{-1}\,,\ d^I_{-2}\}|R_{\bar a}\rangle & \{\alpha^I_{-1}d^J_{-1}\,,\ d^I_{-2}\}|R_a\rangle
\end{array}
$$

Left of bars: $(-1)^F = -1$, the R$-$ sector. Right of bars: $(-1)^F = +1$, the R+ sector.

$$
f_{NS+}(x) = \frac{1}{2\sqrt{x}}\Big[\prod_{n=1}^\infty\Big(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\Big)^8 - \prod_{n=1}^\infty\Big(\frac{1-x^{n-\frac{1}{2}}}{1-x^n}\Big)^8\Big] = 8\prod_{n=1}^\infty\Big(\frac{1+x^n}{1-x^n}\Big)^8 = f_{R-}(x)\,.
$$

Closed strings: When combining left-moving and right-moving sectors we have

$$
\frac{1}{2}\,\alpha' M^2_{\text{closed}} = \alpha' M_L^2 + \alpha' M_R^2\,, \quad \text{with} \quad \alpha' M_L^2 = \alpha' M_R^2
$$